Facultad de Ciencias Económicas y Empresariales
Universidad de Navarra

Working Paper nº 10/05

Linking decisions with moments

Róbert F. Veszteg

Facultad de Ciencias Económicas y Empresariales
Universidad de Navarra

LINKING DECISIONS WITH MOMENTS

Róbert F. Veszteg

Working Paper No.10/05
August 2005
JEL No. C72, D44, D74, D82.

ABSTRACT

This paper proposes a mechanism that can be operated without money in situations where agents have to decide over some common projects when they are not informed about others' preferences. The success of the mechanisms proposed in the literature to deal with similar problems usually relies on the assumption that the entire probability distribution that describes uncertainty is common knowledge. This modified linking mechanism requires the knowledge of solely two moment conditions. It proves to be a useful tool for achieving efficiency improvements in public decision problems. Jackson and Sonnenschein [2005] offer the study of the so-called *linking mechanism*. Here I show that, while allowing for heterogeneity among problems and agents, the linking mechanism keeps its asymptotic properties when run with solely two moment conditions.

Róbert Ferenc Veszteg
Universidad de Navarra, Departamento de Economía
Campus Universitario, Edificio Bibliotecas (Entrada Este)
31080 Pamplona (Navarra), Spain
rveszteg@unav.es

Acknowledgments

I thank Juan Carlos Escanciano, Javier Hualde and David Pérez Castrillo for precious help.

Linking decisions with moments*

Róbert F. Veszteg†

August 2005

Abstract

This paper proposes a mechanism that can be operated without money in situations where agents have to decide over some common projects when they are not informed about others' preferences. The success of the mechanisms proposed in the literature to deal with similar problems usually relies on the assumption that the entire probability distribution that describes uncertainty is common knowledge. This modified linking mechanism requires the knowledge of solely two moment conditions. It proves to be a useful tool for achieving efficiency improvements in public decision problems. Jackson and Sonnenschein [2005] offer the study of the so-called *linking mechanism*. Here I show that, while allowing for heterogeneity among problems and agents, the linking mechanism keeps its asymptotic properties when run with solely two moment conditions.

Keywords: mechanism design, efficiency, public project, uncertainty

JEL Classification Numbers: C72, D44, D74, D82

*I thank Juan Carlos Escanciano, Javier Hualde and David Pérez Castrillo for precious help.

†Universidad de Navarra, Departamento de Economía, Campus Universitario, Edificio Bibliotecas (Entrada Este), 31080 Pamplona (Navarra) - Spain; rveszteg@unav.es.

1 Introduction

Decisions, in which agents hold important private information related to the problem, are modelled as games with incomplete information in the mechanism design literature. Although some important parameters, e.g. private valuations or types, are not known publicly, in Bayesian mechanism design it is usually assumed that the distribution of these parameters is common knowledge.

A large list of mechanisms has been elaborated to cope with this kind of decision problems in order to assure incentive compatibility, and induce truthtelling. One may choose among them taking into account their theoretical properties and/or the assumptions that are required for the results to hold. McAfee [1992], for example, studies the problem of a dissolving partnership in which a set of indivisible objects has to be split between two parties when these are not informed about eachother's valuations. He presents the theoretical properties of several mechanisms, such as the winner's and loser's bid auction or the cake-cutting mechanism, and ranks them according to ex post efficiency.

Jackson and Sonnenschein [2005] offer an interesting proposal for public decision problems that operates without monetary transfers. They prove that the utility costs associated with incentive constraints decrease when the decision problem is linked with independent copies of itself. The linking mechanism, or the social planner that runs it, relies on common knowledge and forces reported valuations to match the commonly known true underlying probability distribution that describes uncertainty in the economy.

Nevertheless, the general assumption of the known underlying distribution is quite demanding in some situations, and therefore mechanisms may fail to extract private information. The planner who runs the linking mechanism may encounter difficulties when announcing the rules of the game, because in order to do so, she must know the entire underlying probability distribution. One may argue that the prior distribution often can be approximated using some finite sample. For example, as a first step, the planner may undertake some (possibly very) costly investigation procedure in order to find out the characteristics of a randomly selected small group of people. Then, in the second step, based on these estimates she can operate some mechanism, e.g., in particular the linking mechanism. However, distributions or high-order moments are difficult to estimate with any reasonable accuracy.

Usually, probability density functions (pdfs) that embody total information about random uncertainties are used to model uncertainty. In most practical real-world applications it is impossible to know or determine the pdf; so, we fall back on using the fact that a pdf is completely characterized by all of its moments. For most pdfs, an infinite number of moments are required.

In this paper, I propose a modified linking mechanism that relies on two moment conditions rather than on all the moment; i.e., on the probability density function. The *linking mechanism with moments* operates in situations in which a series of 0/1 type public decisions have to be made. *Result 1* can be interpreted as carrying out the discussed public project, while *result 0* would mean to maintain the status quo. Uncertainty in this example appears if it is not known publicly how effected parties value each and every public project. The *linking mechanism with moments* requires that individual reports match the first two central moment conditions. Using exclusively the first condition, the mean of the reported types equal to some constant, without side payments would help to reveal whether an agent prefers building the project to the status quo or not, but it would not give reliable information on the intensity of private valuations. Agents would tend to exaggerate on the positive (or negative) consequences of the projects and report the largest (or smallest) admissible valuation. Fixing the second central moment of individual reports overcomes this problem, and allows for aggregation and interpersonal comparison: a project is carried out if the sum of individual reports for it is positive. Also, in practice the first two moment conditions are frequently used to describe a distribution. Note that if all the possible moment conditions are required to hold, we are dealing with the continuous version of the linking mechanism proposed by Jackson and Sonnenschein [2005].

Since the number of decisions linked together would be limited in practice the linking mechanism can not induce truthtelling in general. Moreover, the *linking mechanism with moments* merely imposes two moment conditions, and agents who participate in it can choose their reports from a larger set of possibilities. Since the mechanism imposes fewer conditions this set is larger than the set of admissible reports in the continuous version of the linking mechanism. I show that under some conditions approximately truthful equilibria exist (equilibria in which agents choose the closest admissible reports to their true valuations). In particular, as the number of decisions linked together and the number of participants increases, agents tend to choose their reports approximately truthfully. The rate of convergence of the *linking mechanism with moments* is equal to the one reported by Jackson and Sonnenschein [2005].

Using only two moment conditions makes the mechanism useful in a wide family of decision problems. It allows for heterogeneity both among decision problems and agents, since a large family of different probability distributions can match two given moment conditions. My results, based on different versions of the central limit theorem, show that under some standard assumptions heterogeneity does not spoil approximate truthtelling and efficiency. Throughout the paper, Monte Carlo simulations offer numerical data on the proportion of (ex post) efficient decisions.

The linking mechanism and its version with moments can be related to a series of mechanisms. For example, to the alternating selection mechanism studied by McAfee [1992] in which two parties of a dissolving partnership take turns choosing among a set of indivisible objects to be splitted up. In this case, the two alternatives in each decision problem are *assign a given object to one agent* or *assign it to the other*. Note that if there are only two problems linked together, two moment conditions restrict the set of admissible report to a pair, hence the *linking mechanism with moments* works as a voting scheme. In case of more than two problems one can see the mechanism as similar to the storable votes model proposed by Casella [2003].

Section 2 defines formally the *linking mechanism with moments*. Sections 3 through 5 present its theoretical properties, and finally, section 6 proposes the use of the *linking mechanism with moments* for an empirical problem thas has received the attention of academic and industry researchers lately. Following the list of mechanisms studied by Erev et al. [2004], I argue that assigning the right of way in air traffic in order to avoid conflicts by the linking mechanism may increase efficiency when compared to the actual practice. For more examples and discussion on the related refer to Jackson and Sonnenschein [2005].

2 Linking with moments

Consider a set of agents who have to decide simultaneously over a number of 0/1 type decision problems linked together. In this section I shall consider a situation with m decision problems and n agents whose valuations are modelled as independent random draws from the same underlying probability distribution. This kind of symmetry is assumed for simplicity in this section. Asymmetries are discussed later, in a separate section on heterogeneity. Suppose that the vector $x^i = [x^i_1, x^i_2, \dots x^i_m]$ represents how agent i values decision 1 in each decision problem; i.e., if project j is carried out agent i experiences a utility (increase) of x^i_j units. The 0 decision represents the status quo, and its value is now normalized to 0.[1]

Suppose that agents in the economy hold private information; or in other words, that x^i is not known publicly, but everybody knows that its values vary around a given central parameter, μ, with a fixed variance, σ^2. Technically speaking, I assume that x^i_j is a continuous random variable with $E\left(x^i_j\right) = \mu$ and $Var\left(x^i_j\right) = \sigma^2$ for all i and j, and that μ and σ^2 are common knowledge. In order to reach social decisions, agents will be asked to participate in a mechanism that in what follows I shall refer to as the *linking mechanism with moments*:

Strategies Every agent is asked to choose a vector $y^i = [y^i_1, y^i_2, \dots y^i_m]$ such that two moment conditions hold: $\frac{1}{m}\sum_{j=1}^{m} y^i_j = \mu$ and $\frac{1}{m}\sum_{j=1}^{m}\left(y^i_j - \mu\right)^2 = \sigma^2$.

Payoffs In problem j the final decision is 1 (i.e., project j is carried out), whenever $\sum_{i=1}^{n} y^i_j \geq 0$. In this case agent i enjoys a pay-off of x^i_j, that otherwise would be equal to 0.

This mechanism is closely related to the linking mechanism proposed in Jackson and Sonnenschein [2005]. The latter operates in the discrete case (though it is able to approximate the continuous case), in situations in which the whole prior distribution that characterizes uncertainty is assumed to be common knowledge (i.e. all the moments of the underlying distribution are known). The *linking mechanism with moments* coincides with Jackson and Sonnenschein's proposal on the limit: when the number of applied moment conditions goes very large (infinite).

Agents, who participate in the decision making through the *linking mechanism with moments*, are assumed to choose their reports as to maximize their expected pay-off, hence they solve the following mathematical problem where $M1$ and $M2$ are the two moment conditions:

[1] In other words, the vector x_i represents the difference between agent i's valuations for the two projects.

$$\max_{\{y^i_j\}_{j=1}^m} \sum_{j=1}^m x^i_j \Pr\left(y^i_j > -\sum_{k\neq j} y^k_j\right)$$

$$\text{s.t. } \frac{1}{m}\sum_{j=1}^m y^i_j = \mu, \text{ and} \tag{M1}$$

$$\frac{1}{m}\sum_{j=1}^m \left(y^i_j - \mu\right)^2 = \sigma^2, \tag{M2}$$

with $\mu, \sigma^2 < \infty$. Since private valuations are treated as continuous random variables now, ties are low probability events, because $\Pr\left(y^i_j = -\sum_{k\neq j} y^k_j\right) = 0$. In practice they can be broken by some random device. The notation can be simplified by writing $F\left(y^i_j\right)$ instead of $\Pr\left(y^i_j > -\sum_{k\neq j} y^k_j\right)$.

Even though the social planner's principal objective in order to reach ex post efficient decisions is to extract the private information owned by the agents in the economy, truth might not be an admissible message in the *linking mechanism with moments*, since true valuations might not meet exactly the moment conditions. Nevertheless, by the law of large numbers, as the number of decisions grows, the absolute difference between the two sides of the moment conditions diminishes, when agents report their true valuations. Taking part in the *linking mechanism with moments* necessarily implies lies, therefore I concentrate on approximately truthful strategies instead of truthful ones.

Definition 1. *A message (strategy)* $\widetilde{y}^i = [\widetilde{y}^i_1, \widetilde{y}^i_2, \ldots \widetilde{y}^i_m]$ *is said to be approximately truthful whenever it fulfills the two moment conditions of the mechanism, and its distance from the vector of true valuations,* $x^i = [x^i_1, x^i_2, \ldots x^i_m]$*, is the smallest. Formally, when*

$$\widetilde{y}^i \in \arg\min_{\{y^i_j\}_{j=1}^m} \sum_{j=1}^m \left(y^i_j - x^i_j\right)^2 \textit{ s.t. } \frac{1}{m}\sum_{j=1}^m y^i_j = \mu \textit{ and } \frac{1}{m}\sum_{j=1}^m \left(y^i_j - \mu\right)^2 = \sigma^2.$$

Expected utility maximization and therefore Bayes-Nash equilibria in some cases may result in messages that are far from being truthful. I shall call a Bayes-Nash equilibrium approximately truthful if it involves approximately truthful strategies. The following proposition delivers the condition that guarantees the existence of such an equilibrium for the *linking mech-*

anism with moments.

Proposition 1. *There exists an approximately truthful equilibrium of the linking mechanism with moments, if the distribution function, F, that characterizes uncertainty is linear, or with other words, if $-\sum_{k\neq i} y_j^k$ has uniform distribution.*

Proof. Let $F(x) = \frac{x-a}{b-a}$ for some $a, b \in \mathbb{R}$ such that $a < b$. Now the function to be maximized in the expected utility maximization program can be written as:

$$\sum_{j=1}^{m} x_j^i F\left(y_j^i\right) = \sum_{j=1}^{m} x_j^i \frac{y_j^i - a}{b-a} = \frac{1}{b-a} \sum_{j=1}^{m} \left(x_j^i y_j^i - x_j^i a\right) = const. \cdot \sum_{j=1}^{m} \left(x_j^i y_j^i - const.\right)$$

Therefore the optimal message can be found by maximizing $\sum_{j=1}^{m} x_j^i y_j^i$ over the set defined by the two moment conditions. Let us consider the distance minimization problem:

$$\sum_{j=1}^{m} \left(y_j^i - x_j^i\right)^2 = \sum_{j=1}^{m} \left[\left(y_j^i\right)^2 + \left(x_j^i\right)^2 - 2x_j^i y_j^i\right].$$

Note that the second moment condition can be transformed as

$$\sum_{j=1}^{m} \left(y_j^i - \mu\right)^2 = \sum_{j=1}^{m} \left[\left(y_j^i\right)^2 + \mu^2 - 2\mu y_j^i\right] = \sum_{j=1}^{m} \left(y_j^i\right)^2 - m\mu^2 = m\sigma^2.$$

Therefore $\sum_{j=1}^{m} \left(y_j^i\right)^2 = const.$, such as $\sum_{j=1}^{m} \left(x_j^i\right)^2 = const.$ This means that the distance minimization problem can be transformed into the utility maximization problem and vice versa. □

With the help of the above result it can be shown that as the number of decisions linked together increases the vector of true valuations will be the closest point that fulfills the two moment restrictions. As for the rate of convergence, for the sample mean as an estimator of the expected value, $\overline{y_i} = O_p\left(n^{-\frac{1}{2}}\right)$ holds, while for the variance we similarly have that $\left(s_i^*\right)^2 = O_p\left(n^{-\frac{1}{2}}\right)$. [2] It is important to point out that for the above result (that applies the Euclidean distance) to hold the imposed moment conditions must involve the first two moments.

[2] For example, the book by Davidson and MacKinnon [1993] discusses these convergence concepts and offers an introduction to asymptotic theory.

The question of when the linearity of F is a reasonable assumption still remains open. The following section discusses this problem, and argues that this is the case whenever the number of agents (and with it the degree of the aggregated uncertainty) is very large.

3 Large economies

This section studies whether rational agents may consider F a linear function, or equivalently, consider the corresponding density f constant. Let us apply the following measure for the goodness of the mentioned approximation; i.e., the first order Taylor approximation around μ:

$$Tol = \max_{y \in [y_{\min}; y_{\max}]} \left| f(y) - P_1(y, \mu) \right|.$$

Note that even though individual messages must meet the two moment conditions, the support of the admissible messages might expand. This means that as the number of decisions grows, $m \to \infty$, one can have $y_{\min}$ and/or $y_{\max} \to \infty$. Also note that, as the number of decision problems linked together increases, the absolute value of the largest and smallest admissible message ($y_{\min}$ and $y_{\max}$) also increases. The following proposition states that as the economy gets larger; i.e., as the number of effected parties and/or uncertainty grows, participants of the *linking mechanism with moments* will tend to apply approximately truthful strategies in equilibrium.

Proposition 2. *Suppose that agents approximate the density of the others' aggregated messages with a constant, and the expected value of the underlying probability distribution is zero.*
- If the number of agents and/or the uncertainty grow(s) beyond any limit, the tolerance functions (i.e., the error made in the approximation) goes to zero.
- As for the rate of the convergence to zero, if $y_{\min}$ *or* $y_{\max} = o\left[(n-1)^{-\frac{1}{2}}\right]$*, then* $Tol = o\left[(n-1)^{-\frac{1}{2}}\right]$*. Otherwise* $Tol = O\left[(n-1)^{-\frac{1}{2}}\right]$.

Proof. Without loss of generality let us consider agent 1 as playing *against* the other $(n-1)$ agents in the economy. If $y_j^i \sim iiF$ with expected value $\mu = 0$ and variance σ^2, once we suppose that F has uniformly bounded third moments, we get that $-\sum_{i=2}^{n} y_j^i \sim^a N\left[0; (n-1)\sigma^2\right]$. Moreover, Berry [1941] shows that the error term of this approximation in the neighborhood of zero is of order $n^{-\frac{1}{2}}$. Note that messages sent by an agent are not independent since they are required to meet two moment conditions, but the above sum is over agents and not projects. This is why the central limit theorem applies. With this specification the problem looks as follows:

$$
\begin{aligned}
f(y) &= \frac{1}{\sigma\sqrt{2\pi(n-1)}}\exp\left(-\frac{y^2}{2(n-1)\sigma^2}\right), \\
P_1(y,\mu) &= f(\mu)+f'(\mu)(y-\mu)= \\
&= \frac{1}{\sigma\sqrt{2\pi(n-1)}}\exp\left[-\frac{\mu^2}{2(n-1)\sigma^2}\right]-\frac{1}{\sigma\sqrt{2\pi n}}\frac{2\mu}{2n\sigma^2}\exp\left[-\frac{\mu^2}{2(n-1)\sigma^2}\right](y-\mu), \\
P_1(y,0) &= \frac{1}{\sigma\sqrt{2\pi(n-1)}}, \\
Tol &= \max_{y\in[y_{\min};y_{\max}]}\left|\frac{1}{\sigma\sqrt{2\pi(n-1)}}\exp\left[-\frac{y^2}{2(n-1)\sigma^2}\right]-\frac{1}{\sigma\sqrt{2\pi(n-1)}}\right|= \\
&= \frac{1}{\sigma\sqrt{2\pi(n-1)}}\max_{y\in[y_{\min};y_{\max}]}\left|\exp\left[-\frac{y^2}{2(n-1)\sigma^2}\right]-1\right|.
\end{aligned}
$$

The maximization problem involves a continuous function, and note that $-\frac{1}{2e} < (n-1)\sigma^2$ always holds in the model. For this reason the tolerance function can be simplified:

$$Tol\left(n;\sigma^2\right)=\frac{1}{\sigma\sqrt{2\pi(n-1)}}\max_{y\in[y_{\min};y_{\max}]}\left\{1-\exp\left[-\frac{y^2}{2(n-1)\sigma^2}\right]\right\}.$$

Note that the maximization problem is solved at one of the extremes of the support. Without loss of generality suppose that it happens at y^*.

$$Tol\left(n;\sigma^2\right)=\frac{1}{\sigma\sqrt{2\pi(n-1)}}\left\{1-\exp\left[-\frac{y^{*2}}{2(n-1)\sigma^2}\right]\right\}.$$

Indepedently whether y^* is finite or infinite, the following limit properties can be shown:

$$\lim_{n\to\infty} Tol\left(n;\sigma^2\right)=0,\ \lim_{\sigma^2\to\infty} Tol\left(n;\sigma^2\right)=0 \text{ and } \lim_{n\sigma^2\to\infty} Tol\left(n;\sigma^2\right)=0.$$

Let us check what we can state about the rate of convergence. If y^{*2} is a finite constant, or tends to infinity at a smaller rate than n, then $Tol\left(n;\sigma^2\right) = o\left(n^{-\frac{1}{2}}\right)$, because

$$\lim_{n\to\infty} Tol\left(n;\sigma^2\right) = \lim_{n\to\infty} \frac{Tol\left(n;\sigma^2\right)}{g\left(n\right)} = \lim_{n\to\infty} \frac{\frac{1}{\sigma\sqrt{2\pi(n-1)}}\left[1-\exp\left(-\frac{y^{*2}}{2(n-1)\sigma^2}\right)\right]}{\frac{1}{\sqrt{n-1}}} = 0.$$

If the admissible messages can be any large, and y^{*2} tends to infinity faster than n, what we can state is that $Tol\left(n;\sigma^2\right) = O\left[(n-1)^{-\frac{1}{2}}\right]$. Altogether we have shown that the error term in the approximation around zero of the density of a sum of centered iid random variables with a constant is of order $(n-1)^{-\frac{1}{2}}$, if messages are bounded. Otherwise, it is of order $O\left[(n-1)^{-\frac{1}{2}}\right]$. □

4 Efficiency

By now it has been shown that as the number of participants in the linking mechanism with moments grows, the Bayes-Nash equilibria of the game tend to be approximately truthful. Moreover, as the number of problems linked together increases, approximately truthful equilibria tend to be truthful. Therefore, the *linking mechanism with moments* delivers ex post efficient public decisions if the number of players and the number of decisions linked together tend to infinity.

Proposition 3. *The* linking mechanism with moments *is asymptotically efficient.*

Proof. This result is a direct consequence of the previous proposition and Theorem 1 in Jackson and Sonnenschein [2005]. The latter states that given a decision problem and an ex ante Pareto efficient social choice function, there exist a sequence of linking mechanism on the linked version of the decision problem such that, (1) a corresponding sequence of approximately truthful Bayesian equilibria exists, and (2) the sequence of linking mechanism with these corresponding equilibria approximate the ex ante Pareto efficient social choice function. □

Note that if the *linking mechanism with moments* required all the moment conditions to hold, we would be facing the problem discussed in Jackson and Sonnenschein [2005]. Since I cannot give the explicit formula for computing the number of efficient decisions, I have performed several Monte Carlo experiments in order to express numerically the asymptotic properties of the *linking mechanism with moments*.

The situations I have simulated are the ones that arise when agents report their valuations in an approximately truthful manner. Even though this happens only with a large number of participants, or under some very special assumptions on uncertainty, I report results treating the number of agents as a variable in the analysis.[3] This helps to complete the picture.

4.1 The uniform case

In this exercise I assume that individual private valuations are generated by a random process, precisely that they are drawn from the $[-1;1]$ interval, and each outcome has the same likelihood. Technically: $x_j^i \sim U[-1;1]$ for all i and j. The problem that individual messages solve in the approximately truthful case is the following for every agent i:

$$\min_{\{y_j^i\}_{j=1}^m} \sum_{j=1}^m \left(x_j^i - y_j^i\right)^2 \text{ s.t. } \frac{1}{m}\sum_{j=1}^m y_j^i = 0 \text{ and} \tag{M1}$$

$$\frac{1}{m}\sum_{j=1}^m \left(y_j^i\right)^2 = \frac{1}{3}. \tag{M2}$$

Let us solve the problem and express the solution as a system of equations.

$$\begin{aligned} L &= \sum_{j=1}^m \left(x_j^i - y_j^i\right)^2 - \frac{\lambda_1}{m}\sum_{j=1}^m y_j^i - \lambda_2\left[\frac{1}{m}\sum_{j=1}^m \left(y_j^i\right)^2 - \frac{1}{3}\right] \\ \frac{\partial L}{\partial y_j^i} &= -2\left(x_j^i - y_j^i\right) - \frac{\lambda_1}{m} - \frac{2\lambda_2}{m} y_j^i = 0 \end{aligned}$$

In order to eliminate the Lagrange-multipliers from the above system consider the equations for $j = 1$ and 2.

$$\begin{aligned} -2\left(x_1^i - y_1^i\right) - \frac{\lambda_1}{m} - \frac{2\lambda_2}{m} y_1^i &= 0 \implies \lambda_1 = -2m\left(x_1^i - y_1^i\right) - 2\lambda_2 y_1^i \\ -2\left(x_2^i - y_2^i\right) - \frac{\lambda_1}{m} - \frac{2\lambda_2}{m} y_2^i &= 0 \implies \lambda_2\left(y_2^i - y_1^i\right) = m\left[\left(x_1^i - y_1^i\right) - \left(x_2^i - y_2^i\right)\right] \end{aligned}$$

[3] As shown in the previous section, for approximate truthful messages, the distribution of the others' aggregated bid is required to be uniform.

If $x_2^i = x_1^i$, then $y_2^i = y_1^i$ (but this event happens with probability zero). In any other case $\lambda_2 = \frac{m\left[\left(x_1^i - x_2^i\right) - \left(y_1^i - y_2^i\right)\right]}{y_2^i - y_1^i}$, and $\lambda_1 = -2m\left(x_1^i - y_1^i\right) - 2my_1^i \frac{\left[\left(x_1^i - x_2^i\right) - \left(y_1^i - y_2^i\right)\right]}{y_2^i - y_1^i}$. Then for $j = 3, 4, ..., m$:

$$-2\left(x_j^i - y_j^i\right) - \frac{\lambda_1}{m} - \frac{2\lambda_2}{m} y_j^i = 0 \implies y_j^i = \frac{2mx_j^i + \lambda_1}{2\left(m - \lambda_2\right)}.$$

Altogether the following system should be solved:

$$y_1^i + y_2^i + y_3^i + ... + y_m^i = 0, \tag{M1}$$

$$y_1^i + y_2^i + \frac{2mx_3^i + \lambda_1}{2\left(m - \lambda_2\right)} + ... + \frac{2mx_m^i + \lambda_1}{2\left(m - \lambda_2\right)} = 0, \tag{M1}$$

$$\left(y_1^i\right)^2 + \left(y_2^i\right)^2 + \left(y_3^i\right)^2 + ... + \left(y_m^i\right)^2 = \frac{m}{3}, \tag{M2}$$

$$\left(y_1^i\right)^2 + \left(y_2^i\right)^2 + \left(\frac{2mx_3^i + \lambda_1}{2\left(m - \lambda_2\right)}\right)^2 + ... + \left(\frac{2mx_m^i + \lambda_1}{2\left(m - \lambda_2\right)}\right)^2 = \frac{m}{3}. \tag{M2}$$

The following table reports the number of efficient decisions for different values of m (number of decisions linked together) and n (number of agents). The computation has been performed with GAUSS6.0 and 5'000 iterations have been made in each case. The last column, for $n = 50$, can be interpreted as the one that approximates the theoretical case of large economies. As predicted by Proposition **??** the proportion of ex post efficient decision is an increasing function of the number of decision problems linked together.

m ╲ n	2	5	10	20	50
2	67%	70%	69%	70%	70%
5	83%	84%	83%	83%	83%
10	89%	84%	89%	88%	89%
20	92%	92%	92%	92%	92%
50	95%	95%	95%	95%	95%

Table 1. Proportion of ex post efficient decisions in the U$[-1;1]$ case.

4.2 Other distributions

Table 2 and 3 report result with similar pattern for different distributions. The standard normal case is important, because it uses a distribution that play important role in theory, while the χ^2 distribution has been chosen because of its asymmetry. The latter has been transformed in order to have a centered random variable that is required for the results in this model.

m $\diagdown$ n	2	5	10	20	50
2	66%	69%	68%	68%	67%
5	83%	82%	81%	81%	81%
10	88%	87%	87%	86%	86%
20	92%	91%	90%	90%	90%
50	95%	94%	94%	94%	94%

Table 2. Proportion of ex post efficient decisions in the $\chi^2(5)$case.

m $\diagdown$ n	2	5	10	20	50
2	67%	71%	69%	69%	69%
5	83%	83%	82%	82%	82%
10	88%	88%	88%	87%	87%
20	92%	92%	91%	91%	91%
50	95%	95%	95%	95%	95%

Table 3. Proportion of ex post efficient decisions in the $N(0;1)$case.

5 Individual rationality

Individual rationality refers to participation constraints. A mechanism is called individually rational if agents prefer to take part in it to asbtain from it. We talk about ex ante, interim and ex post individual rationality depending on at which point of time is the participation decision concerned.

As discussed in Jackson and Sonnenschein [2005], in the linking mechanism the ex ante expected utility level taking into account the true private valuations; i.e., $E\left[u_i\left[g\left(x_j\right)\right]\right] = u_j^*$, is reached on the limit that translates in our case into a situation in which the number of agents and the number of decision problems linked together grow beyond all limit. The function g represents an ex post efficient social choice function.

In spite of this feature of the linking mechanism agents may want to abstain from participation, because the moment conditions imposed on reported valuations force them to pronounce clearly in favor of some projects and against the others. Although not likely, but an agent might happen to value negatively all the proposed projects and therefore prefer the status-quo. Similarly to the linking mechanism discussed in Jackson and Sonnenschein [2005], the *linking mechanism with moments* can be modified in order to induce individual rationality. For this a second stage should be added to the original mechanism in which agents decide whether they wish to participate or not. If any of them decides not to, then no social decision is made and the status-quo is maintained. Jackson and Sonnenschein [2005] show that the modified version of the linking mechanism satisfies the ex post (with that also the interim and ex ante) participation constraint. Since their argument holds in my set-up too, the proof is omitted here.

6 Heterogeneity

It is worth noting that the result for large economies holds in cases in which the distribution of private valuations, F, differs across problems; i.e., when dealing with F_js. This means that one can link different decisions problems together without losing the appealing theoretical properties of the mechanism, as long as the underlying distributions behind the decision problems are independent and not different in their first two central moments.

Table 4 reports the results of a Monte Carlo experiment that simulates an economy with two types of decision problems. One type has attached private valuations that are drawn from a symmetric normal distribution, while the other are generated by an asymmetric χ^2 distribution. Types have been assigned in a random manner: a given decision problem has the same probability to be of type normal and type χ^2. The random variables have been centered, and transformed in order to have the same second moment. As predicted by theory, the pattern that can be observed in Tables 1-3 is not altered by the presence of heterogeneity of the decision problems.

Not only identical problems might be difficult to encounter in order to link them together and improve on ex post efficiency of the social decisions, but also effected parties, i.e., participants in the linking mechanism with moments, might differ from each other. Fortunately, this heterogeneity across agents does not necessarily invalidate my results.

The main result for large economies, the linearity of F on the limit, is based on the central limit theorem that, in its classical form, holds for independent, identically-distributed random variables. By Lindeberg's theorem the assumption of identical distribution can be dropped and the limiting distribution of the sum will still be the normal, as long as the so-called Lindeberg condition holds.[4] Naturally, the linking mechanism with moments does not allow for all kind of heterogeneity among agents, as it requires that the underlying distributions meet two moment conditions and have finite variance. Precisely this makes the Lindeberg condition and with it my results hold also under this type of heterogeneity.[5] Table 5 report results from the simulations according to which both agents and decision problems are heterogeneous: with probability $\frac{1}{2}$ the private valuation x_j^i is drawn from the $\chi^2(5)$distribution, and with probability $\frac{1}{2}$, it is a realization of the $N(0;10)$ distribution. Note that the proportion of ex post efficient decisions is barely affected by the heterogeneity.

m ╲ n	2	5	10	20	50
2	66%	70%	67%	67%	68%
5	83%	82%	82%	81%	80%
10	89%	88%	87%	86%	85%
20	92%	91%	91%	91%	90%
50	95%	94%	94%	94%	94%

Table 4. Proportion of ex post efficient decisions in the heterogeneous problems [$\chi^2(5)$&$N(0;10)$]case.

m ╲ n	2	5	10	20	50
2	66%	70%	67%	69%	70%
5	83%	82%	82%	81%	82%
10	88%	88%	87%	87%	86%
20	92%	92%	91%	91%	90%
50	95%	95%	94%	94%	94%

Table 5. Proportion of ex post efficient decisions in the heterogeneous problems and agents [$\chi^2(5)$ & $N(0;10)$]case.

I have argued before that in practice agents and the central planner are likely to have precise information only on the first two central moments of the underlying distributions (means and

[4] See, for example, Billingsley [1995] for details.

[5] In the model of the *linking mechanism with moments*, the Lindeberg condition can be simplified to:

$$\lim_{n\to\infty} \frac{1}{n\sigma^2} \int_{\left|x_j^i\right|>\varepsilon\sigma\sqrt{n}} \left(x_j^i\right)^2 dF_j^i = 0 \text{ for all } i \text{ and } j.$$

It is not difficult to show that the above condition is satisfied in this model, and therefore Lindeberg's theorem holds.

variances). After considering the issue of heterogeneity in the model, a short comment is in order: there exist a very large number of distributions that have some given expected value and variance, therefore uncertainty is very complicated to perceive and also to model. In these situations uninformative (uniform) priors could be a reasonable choice by participants. As discussed before, the *linking mechsnism with moments* with uniform priors leads to an equilibrium with approximate truthtelling.

7 Practical example: the right of way in the sky[6]

Recent technological and regulatory developments in air transportation have raised a debate about who is responsible for the adequate separation between aircrafts in the air and who should maneuver to avoid conflicts. Current practice relies on a centralized system in which air traffic controllers execute this task. The changes under considerations allow for *free flight,* and opt for decentralization by proposing the idea of self-separation between aircrafts. According to the latter, right-of-way rules should resolve conflicts and determine who has to undergo an avoiding maneuver. The existing rules state that the right of way goes to the aircraft that comes from the right or is in front, in case of overtaking. As argued by Erev et al. [2004] efficient right-of-way rules should take into account both technological and economic constraints, and also consider strategic behavior among the potential parties of a conflict in the air. Current regulation gives way to traffic coming from the right, while economic efficiency requires giving it to the party that values it most. Considering both rational and boundedly rational agents, Erev et al. [2004] suggest a series of possible *rules* from the mechanism design literature based on their theoretical properties. I believe that the linking mechanism with moments is a suitable candidate to be included in this list. Its appealing theoretical properties have been shown in this paper. It is simple, yet applicable in a variety of situations without the necessity of monetary side payments. The lack of money eliminates the incentives to initiate conflict, and as discussed below in this section, this mechanism increases efficiency in the conflict resolution.

The *linking mechanism with moments* can be tailored for this problem as follows. Airlines

[6] The problem discussed in this section is considered in detail by Erev et al. [2004]. They propose and discuss a series of feasible solutions that the economic literature offers at the moment, such as the alternating offers mechanism, negotiation with side payments and the sealed bid auction.

(the affected parties or players in this game) are periodically asked to attach a real number to every flight due to take-off before the next reporting time. The length of the period between reports should be determined as to ensure feasibility, relatively little operational cost to airlines and a sufficient number of flights to be considered simultaneously, since the appealing properties of the mechanism require a large number of simultaneous reports from every player. The number attached to flight j by airline i in period t is $y_j^{it} \in \mathbb{R}$. According to the rules of the *linking mechanism with moments* these numbers should meet two moment conditions in every period, and in equilibrium they reflect how airlines value the right of way (the importance of being on time at the destination) of the given flight.[7] Theoretical moments, in order to fix the restrictions, can be defined by previously conducted throughout statistical surveys for every reporting period. Now, when two aircrafts enter in conflict the one with the higher attached value receives the right of way, while the other must undergo an avoiding maneuver. Ties, although being zero probability events in theory, can be resolved by a random device.

Proposition **??** and the numerical results in Tables 1 through 5 are the theoretical proofs for how this mechanism would be able to improve efficiency in the right-of-way problem, whenever the assumptions of the model hold.

8 Conclusions

The *linking mechanism with moments* has been presented for public decision problems. It is a less demanding version of the linking mechanism from an informational point of view, nevertheless it keeps its asymptotic properties. The simpleness of its rules, the feature that it operates without monetary transfers and its intuitive equilibria make the mechanism attractive for applications, although little is known about its performance in the finite world. The characterization of its equilibria with a small number of participants may also help to explore whether the linking mechanism is immune to coalitional deviations. The argument presented by Jackson and Sonnenschein [2005] does not apply directly to this case, because coalition formation reduces the number of participants.

The similarities and differences between using the linking mechanism and the *linking mech-*

[7] For airlines with only one flight in some period, only the first moment condition is required to hold.

anism with moments can also be represented by a parallelism from econometrics. The maximum likelihood estimator assumes that the whole underlying probability distribution is known, while the method of moments matches only a set of empirical moments with theoretical ones. As for their asymptotic properties, under specific conditions, both estimation procedures deliver consistent estimates at the same rate of convergence (if n denotes the number of observations: $\frac{1}{\sqrt{n}}$).[8] There are no general results on the small sample properties of these estimators. These topics are still objects of research.

Decision problems in the linking mechanism are assumed to be independent, and major results are based on central limit theorems designed for the case of independent random variables. However, a sufficiently large number of independent decision problems may be difficult to find. The performance of the linking mechanism with correlated types could be an important topic for future studies.

References

[1] Berry, A.C. (1941). The Accuracy of the Gaussian Approximation to the Sum of Independent Variables. Transactions of the American Mathematical Society. 1: 122-136.

[2] Billingsley, P. (1995). Probability and measure (Wiley series in probability and mathematical statistics). Third edition. A Wiley-Interscience publication.

[3] Casella, A. (2003). Storable Votes. mimeo: Columbia University.

[4] Davidson, R. and MacKinnon, J. (1993) Estimation and Inference in Econometrics. Oxford University Press, USA.

[5] Erev, I., Barron, G. and Remington, R. (2004) Right of Way in the Sky: Two Problems in Aircraft Self-Separation and the Auction-Based Solution. Human Factors. 2: 267-276.

[6] Jackson, M. and Sonnenschein, H.F. (2005) Overcoming Incentive Constraints by Linking Decisions. Caltech Working Paper. Forthcoming: Econometrica

[8] For precise discussion refer, e.g., to van der Vaart [1998].

[7] McAfee, R. P. (1992). Amicable divorce: dissolving a partnership with simple mechanisms. Journal of Economic Theory. 56: 266-293.

[8] van der Vaart, A.W. (1998). Asymtotic statistics (Cambridge series in statistical and probabilistic mathematics). Cambridge University Press.

[9] Veszteg, R.F. (2004) Multibidding Game under Uncertainty. Working Paper 14/04. Universidad de Navarra.